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# ON STABILITY OF MECHANICAL SYSTEMS UNDER THE ACTION OF POSITION FORCES *

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The stability problem is solved for the equilibrium position of a holonomic mechanical system subject to stationary geometric constraints and to potential and nonconservative position forces /1/. It is assumed that the characteristic equation of the linear approximation has pure imaginary roots among which there are none equal. The system being examined is invertible and is Birkhoff-stable/3/ when the system does not have an internal resonance in the sense of /2/: finite-order instability can be detected only under internal resonance. Necessary and sufficient stability conditions for a model system and sufficient Liapunov-instability conditions have been formulated for odd-ordered resonances. A fourth-order resonance, of greatest importance among even-ordered resonances for applications, has been investigated and for it necessary and sufficient stability conditions in the first nonlinear approximation have been obtained in the absence of degeneracy; it is shown that Liapunov-instability follows from third-order instability. An example is presented.

**1. Statement of the problem.** We consider a holonomic mechanical system with $n$ degrees of freedom, subject to stationary geometric constraints. If as the basic variables characterizing the system's state at any instant $t$ we take the independent Lagrange variables $q_s$ and velocities $q_s^{\cdot} = dq_s / dt$, then the system's equations of motion can be written as the Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial q_s^{\cdot}} - \frac{\partial T}{\partial q_s} = Q_s(q) \quad (s = 1, \ldots, n), \quad 2T = \sum_{i,j=1}^{n} a_{ij}(q)\, q_i^{\cdot} q_j^{\cdot} \tag{1.1}$$

Here $T$ is the kinematic energy, while the generalized forces $Q_s$ depend only on coordinates $q$; $Q_s$ and $a_{ij}$ are holomorphic functions of $q$. As is well known /1/, any force $Q(q) = (Q_1, \ldots, Q_n)$, continuous together with its first-order derivatives and depending only on the system's position, can be decomposed into potential and nonconservative position components. We assume henceforth that the nonconservative position component of force $Q$ is nonzero. We investigate the stability of the equilibrium position of system (1.1) assuming, without loss of generality, that zero coordinate values $q_s^{\circ} = 0$ and $Q_s(0) = 0$ $(s = 1, \ldots, n)$ correspond to the equilibrium position.

Having solved system (1.1) relative to the highest derivatives, we write it in the form

$$q_s^{\cdot\cdot} = \sum_{i=1}^{n} b_{si} q_i + F_s(q) + \sum_{i,j=1}^{n} c_{sij}(q)\, q_i^{\cdot} q_j^{\cdot} \quad (s = 1, \ldots, n) \tag{1.2}$$

where $F_s$ and $c_{sij}$ are holomorphic functions of $q_1, \ldots, q_n$ and the expansions of functions $F_s$ start with terms of no lower than second order in $q$; $b_{si}$ are constants. The system's characteristic equation

$$\Delta(\varkappa^2) = \det \| b_{si} - \delta_{si}\varkappa^2 \| = 0 \tag{1.3}$$

has only even powers of $\varkappa$, consequently, if among the roots $\varkappa^2 = \lambda_1^2, \ldots, \lambda_n^2$ of (1.3) there is at least one complex or positive root, then the equilibrium position is unstable /1/. This is possible, for instance, in the absence of potential forces if the nonconservative position forces do not equal zero in the linear approximation /1/. Let us assume that $\lambda_s^2 < 0$ $(s = 1, \ldots, n)$. System (1.2) goes into itself under the linear substitution $R: t \to -t$, $q \to q$, $q^{\cdot} \to -q^{\cdot}$, i.e., is invertible (possesses a linear automorphism /4/). Therefore, if in the system there is no internal resonance in the sense of /2/, then the equilibrium position is a point of complete Birkhoff-stability /3,5/.

By examining the linear-approximation system

$$q_s^{\cdot\cdot} = \sum_{i=1}^{n} b_{si} q_i \quad (s = 1, \ldots, n) \tag{1.4}$$

we pass to the complex-conjugate variables $z_s$ and $\bar{z}_s$ $(s = 1, \ldots, n)$ by the linear substitution

$$z_s = \sum_{j=1}^{n} p_{sj}(q_j^{\cdot} + \lambda_s q_j), \quad \bar{z}_s = \sum_{j=1}^{n} p_{sj}(-q_j^{\cdot} + \lambda_s q_j) \tag{1.5}$$

We define the pure imaginary constants $p_{sj}$ such that system (1.4) has the form

$$z_s^{\cdot} = \lambda_s z_s, \quad \bar{z}_s^{\cdot} = -\lambda_s \bar{z}_s \quad (s = 1, \ldots, n) \tag{1.6}$$

in the new variables. Hence, substituting (1.5) into (1.6) and taking into account that $q_s$

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satisfy (1.4), we obtain the system of equations

$$\begin{aligned} &(b_{11}-\lambda_s^2)p_{s1}+b_{21}p_{s2}+\ldots+b_{n1}p_{sn}=0\\ &b_{12}p_{s1}+(b_{22}-\lambda_s^2)p_{s2}+\ldots+b_{n2}p_{sn}=0\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &b_{1n}p_{s1}+b_{2n}p_{s2}+\ldots+(b_{nn}-\lambda_s^2)p_{nn}=0\quad(s=1,\ldots,n)\end{aligned}\tag{1.7}$$

for computing the constants $p_{sj}$ $(s, j=1,\ldots,n)$. Since $\lambda_s^2$ are the roots of characteristic Eq. (1.3), the determinant of system (1.7) equals zero and the latter admits of a nontrivial solution. Obviously, if among the $\lambda_s^2$ there are none equal to one another, we have $\det\|p_{sj}\|\neq 0$, whence immediately follows the nonsingularity of transformation (1.5): the determinant of the matrix of this transformation is

$$2\prod_{j=1}^{n}\lambda_j\{\det\|p_{sj}\|\}^2$$

Let us express $q_s$ and $q_s^{\cdot}$ in terms of the new variables $z_s$ and $\bar z_s$. From (1.6) we have

$$z_s-\bar z_s=2\sum_{j=1}^{n}p_{sj}q_j^{\cdot},\quad z_s+\bar z_s=2\lambda_s\sum_{j=1}^{n}p_{sj}q_j\quad(s=1,\ldots,n)$$

whence (the $d_{sj}$ are pure imaginary constants)

$$q_s=\frac12\sum_{j=1}^{n}\frac{d_{sj}}{\lambda_j}(z_j+\bar z_j),\quad q_s^{\cdot}=\frac12\sum_{j=1}^{n}d_{sj}(z_j-\bar z_j)\quad(s=1,\ldots,n)\tag{1.8}$$

Now we write the result of passing to the new variables as

$$\begin{aligned} &z^{\cdot}=\Lambda z+Z(z,\bar z),\quad \bar z^{\cdot}=-\Lambda\bar z+\bar Z(z,\bar z)\\ &z=(z_1,\ldots,z_n)\quad \bar z=(\bar z_1,\ldots,\bar z_n),\quad Z=(Z_1,\ldots,Z_n)\\ &\bar Z=(\bar Z_1,\ldots,\bar Z_n),\quad \Lambda=\operatorname{diag}(\lambda_1,\ldots,\lambda_n)\end{aligned}\tag{1.9}$$

$$Z_s=\sum_{r=1}^{n}p_{sr}\left[F_r(q)+\sum_{i,j=1}^{n}c_{rij}(q)\,q_i^{\cdot}q_j^{\cdot}\right]\Bigg|_{(1.8)}\tag{1.10}$$

Here $Z$ and $\bar Z$ are complex-conjugate analytic vector-valued functions of $z$ and $\bar z$, $\Lambda$ is the diagonal eigenvalue matrix; the notation in (1.10) signifies that a passage to variables $z$ and $\bar z$ by formulas (1.8) has been effected in the right hand sides. According to (1.8) and (1.10), the expansions of $Z_s$ and $\bar Z_s$ into series in powers of $z$ and $\bar z$ have only pure imaginary coefficients. This fact is a corollary of the presence in the system of the linear automorphism $R\colon t\to -t,\ z\to\bar z,\ \bar z\to z$, which is preserved in the normal form /4/. Henceforth we assume that the normal form of system (1.9), (1.10) has been written in variables $u$ and $v$; $z,\bar z\to u,v$.

**2. The stability problem.** From the above it follows that finite-order instability can be detected only if internal resonance in the sense defined in /2/ is present in the system. The solution of the stability problem has been given in /2/ for autonomous systems of differential equations with respect to the first nonlinear terms in the normal form in the case of internal resonance of odd order $K$. For the problem being examined these results can be stated as follows. In the polar coordinates

$$u_s=\sqrt{r_s}\exp(i\theta_s),\quad v_s=\sqrt{r_s}\exp(-i\theta_s)\quad(s=1,\ldots,n)\tag{2.1}$$

the model system obtained from the normal form by dropping terms of order higher than $K-1$ can be written as

$$\begin{aligned} &r_s^{\cdot}=2a_s\sin\theta\prod_{j=1}^{m}r_j^{p_j/2}\quad(s=1,\ldots,m)\\ &\theta^{\cdot}=\sum_{s=1}^{m}p_sa_s\cos\theta\prod_{j=1}^{m}r_j^{p_j/2-\delta_{sj}},\quad \theta=p_1\theta_1+\ldots+p_m\theta_m,\quad p_s>0\\ &r_\alpha^{\cdot}=0\quad(\alpha=m+1,\ldots,n)\end{aligned}\tag{2.2}$$

where $a_s$ $(s=1,\ldots,m)$ are real constants, and is a certain special class of the systems considered in /2/.

**Theorem 1.** A necessary and sufficient stability condition for model system (2.2) is the existence of a pair of coefficients $a_i, a_j\neq 0$ such that $\operatorname{sign} a_ia_j=-1$. If, however, the condition $\operatorname{sign} a_ia_j=1$ is fulfilled for any pair $a_i, a_j$, then the trivial solution of system (1.1) is Liapunov-unstable.

The question of the stability of the model system when several odd-ordered resonances are present in a system of form (1.9), (1.10) was investigated in /6/. Let us investigate the stability when an even-ordered resonance is present in the system; among them we consider fourth-order resonances as being the most important for applications. As is well know /7/, the stability problem with fourth-order resonance is rather complex and in the general case an algebraic stability criterion does not exist even for the model system (*); we have managed to obtain only certain sufficient conditions for asymptotic stability and instability. However, the question for the model system has been solved to completion for the class of mechanical systems (1.1) being examined; when degeneracy is absent the necessary stability conditions are sufficient as well.

Without loss of generality we write this resonance as

$$\sum_{i=1}^{m} p_i\lambda_i = 0, \quad p_i > 0, \quad \sum_{i=1}^{m} p_i = 4, \quad m \leqslant 4, \quad m \leqslant n \tag{2.3}$$

According to /4/ and to Sect. 1, the model system resulting from the normal form by deletion of terms of higher than third order is

$$\begin{aligned} u_\alpha^{\cdot} &= \lambda_\alpha u_\alpha + iu_\alpha \sum_{j=1}^{n} A_{\alpha j} u_j v_j + iB_\alpha \prod_{j=1}^{m} v_j^{p_j-\delta_{\alpha j}} \\ u_\beta^{\cdot} &= \lambda_\beta u_\beta + iu_\beta \sum_{j=1}^{n} A_{\beta j} u_j v_j \quad (\alpha = 1, \ldots, m;\ \beta = m+1, \ldots, n) \end{aligned} \tag{2.4}$$

Here $A_{\alpha j}$, $A_{\beta j}$, $B_\alpha$ are real constants; the complex-conjugate group of equations has been omitted. We consider the nonsingular case when not even one of the coefficients $B_\alpha$ vanishes. In the polar coordinates $r_s$, $\theta_s$ (from formulas (2.1)) system (2.4) is written as

$$\begin{aligned} & r_\alpha^{\cdot} = 2B_\alpha \sin\theta \prod_{j=1}^{m} r_j^{p_j/2}, \quad r_\beta^{\cdot} = 0 \\ & (\alpha = 1, \ldots, m;\ \beta = m+1, \ldots, n) \\ & \theta^{\cdot} = \sum_{j=1}^{n} A_j r_j + \sum_{k=1}^{m} p_j B_j \prod_{k=1}^{m} r_k^{p_k/2-\delta_{jk}} \cos\theta \\ & A_j = \sum_{\alpha=1}^{m} p_\alpha A_{\alpha j}, \quad \theta = \sum_{\alpha=1}^{m} p_\alpha \theta_\alpha \quad (j = 1, \ldots, n) \end{aligned} \tag{2.5}$$

Direct substitution convinces us that system (2.5) has the following first integrals

$$\begin{aligned} & W_\alpha = B_\alpha r_1 - B_1 r_\alpha = h_\alpha \quad (\alpha = 2, \ldots, m) \\ & W_\beta = r_\beta = h_\beta \quad (\beta = m+1, \ldots, n) \\ & W = \sum_{\alpha=1}^{m} A_\alpha \Big(\prod_{j=1}^{m} B_j^{1-\delta_{\alpha j}}\Big) r_\alpha^2 + 4 \prod_{j=1}^{m} B_j r_j^{p_j/2} \cos\theta + \\ & \quad \frac{4}{m} \sum_{\beta=m+1}^{n} A_\beta r_\beta \Big(\sum_{\alpha=1}^{m} r_\alpha \prod_{j=1}^{m} B_j^{1-\delta_{\alpha j}}\Big) = h \end{aligned} \tag{2.6}$$

where $h_\nu\ (\nu = 2, \ldots, n)$ and $h$ are arbitrary constants. Consequently, if there is a change of sign among the coefficients $B_\alpha\ (\alpha = 1, \ldots, m)$, then from the first and second integrals in (2.6) we can set up a sign-definite integral linear in $r_s\ (s = 1, \ldots, n)$, which proves the stability of model system (2.4).

Suppose that all the $B_\alpha$ are of the same sign. From the integrals (2.6) we form the function

$$V(r_1, \ldots, r_n, \theta) = W^2 + \sum_{\nu=2}^{n} W_\nu^4$$

Function $V$ is obviously positive definite in $r_1, \ldots, r_n$, if $W \neq 0$ on the manifolds

$$W_\nu = 0 \quad (\nu = 2, \ldots, n) \tag{2.7}$$

on (2.7) function $W$ has the form

$$W_* = \Big(\prod_{j=1}^{m} B_j / B_1^2\Big)\Big[\sum_{\alpha=1}^{m} A_\alpha B_\alpha + 4\cos\theta \prod_{j=1}^{m} |B_j|^{p_j/2}\Big] r_1^2$$

and does not vanish if

$$\Big|\sum_{\alpha=1}^{m} A_\alpha B_\alpha\Big| > 4 \prod_{j=1}^{m} |B_j|^{p_j/2} \tag{2.8}$$

Thus, if all $B_\alpha$ are of one sign, then when (2.8) is fulfilled the function $V$ is a Liapunov function for (2.5), satisfying the stability theorem in /8/.

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*) See: Shnol', E. E. and Khazin, L. G., Nonexistence of an algebraic asymptotic stability criterion under resonance 1:3. Preprint Inst. Prikl. Mat., No. 45, Moscow, 1977.

Now let the inequality sign in (2.8) be reversed and let all $B_\alpha$ be of one sign. Then the model system has a particular solution of growing ray type

$$r_\alpha = \gamma_\alpha r, \quad r_\beta = 0, \quad r^\cdot = \gamma r^2, \quad \theta = \theta_0; \quad \gamma, \gamma_\alpha > 0$$
$$(\alpha = 1, \ldots, m; \ \beta = m+1, \ldots, n)$$

Indeed, substituting this solution into (2.5), we obtain

$$r^\cdot = 2r^2\left(\frac{B_\alpha}{\gamma_\alpha}\prod_{j=1}^{m}\gamma_j^{p_j/2}\right)\sin\theta, \ \theta^\cdot = \left[\sum_{\alpha=1}^{m}A_\alpha\gamma_\alpha + 4B_1\prod_{j=1}^{m}\gamma_j^{p_j/2}\cos\theta\right]r$$

whence

$$\gamma_1 = 1, \quad \gamma_2 = \frac{B_2}{B_1}, \ldots, \quad \gamma_m = \frac{B_m}{B_1}, \quad 4\,|\cos\theta_0|\prod_{j=1}^{m}|B_j|^{p_j/2} =$$
$$\left|\sum_{\alpha=1}^{m}A_\alpha B_\alpha\right|, \quad B_1\sin\theta_0 > 0$$

consequently, in this case the model system is unstable. The stability question for the model system has been completely solved. We note that the system of integrals (2.6) is sufficient for the reduction of the model system's integration problem to quadratures.

Let us now show that the instability revealed in the model system is preserved in the complete system. Assuming that system (1.9) has been reduced to normal form up to third-order terms, inclusive, we pass to polar coordinates by formulas (2.1). The result can always be written as

$$r_\alpha^\cdot = 2\,\mathrm{sign}\,B_\alpha\prod_{j=1}^{m}|B_j|^{p_j/2}r_j^{p_j/2}\sin\theta + R_\alpha(r_*, \theta_*) \tag{2.9}$$
$$\prod_{j=1}^{m}r_j^{p_j/2}\theta^\cdot = \prod_{j=1}^{m}r_j^{p_j/2}\Big[\sum_{j=1}^{m}A_j|B_j|r_j + \sum_{\beta=m+1}^{n}A_\beta r_\beta +$$
$$\prod_{j=1}^{m}|B_j|^{p_j/2}\sum_{\alpha=1}^{m}p_\alpha\,\mathrm{sign}\,B_\alpha\prod_{j=1}^{m}r_j^{p_j/2-\delta_{\alpha j}}\cos\theta\Big] + \Phi(r_*, \theta_*)$$
$$r_\beta^\cdot = R_\beta(r_*, \theta_*) \quad (\alpha = 1, \ldots, m^\cdot, \quad \beta = m+1, \ldots, n)$$
$$r_* = (r_1, \ldots, r_n), \quad \theta_* = (\theta_1, \ldots, \theta_n), \quad \|r_*\| = \left(\sum_{i=1}^{n}r_i^{1/2}\right)^2$$
$$\Phi \sim O(\|r_*\|^{3/2}), \quad R_{\alpha,\beta} \sim O(\|r_*\|^{5/2})$$

Now, instead of variables $r_\alpha$ we introduce the following ones

$$r_1, \ r_2 = r_1(1 + x_2), \ \ldots, \quad r_m = r_1(1 + x_m)$$

corresponding to the equations

$$r_1x_\alpha^\cdot = -2r_1^2x_\alpha\,\mathrm{sign}\,B_\alpha|B_1|^{p_1/2}\prod_{j=1}^{m}|B_j|^{p_j/2}(1+x_j)^{p_j/2}\sin\theta + R_\alpha - R_1(1+x_\alpha) \tag{2.10}$$

and we consider the function

$$V = r_1\Big[\pm\sum_{\alpha=1}^{m}r_\alpha\prod_{j=1}^{m}r_j^{p_j/2}\sin\theta - \sum_{\beta=m+1}^{n}r_\beta^{3(1-\gamma)} - \sum_{\mu=2}^{m}x_\mu^{10}\Big]$$
$$(0 < \gamma < 1/3)$$

where the plus sign is selected when all $B_\alpha$ are positive and the minus sign when all $B_\alpha$ are negative. Computing the derivative of $V$ relative to Eqs. (2.9) and (2.10), in the domain $V > 0$ we obtain

$$V^\cdot = \pm 2\prod_{j=1}^{m}|B_j|^{p_j/2}r_j^{p_j/2}\sin\theta\Big[\pm\sum_{\alpha=1}^{m}r_\alpha\prod_{j=1}^{m}r_j^{p_j/2}\sin\theta -$$
$$\sum_{\beta=m+1}^{n}r_\beta^{3(1-\gamma)} - \sum_{\mu=2}^{m}x_\mu^{10}\Big] +$$
$$m\Big[2\prod_{j=1}^{m}|B_j|^{p_j/2}\sin^2\theta + 4\prod_{j=1}^{m}|B_j|^{p_j/2} + \sum_{\alpha=1}^{m}A_\alpha|B_\alpha|\cos\theta\Big]r_1^5 \pm$$
$$20r_1^2\sum_{\mu=2}^{m}|B_1|^{p_1/2}x_\mu^{10}\prod_{j=2}^{m}|B_j|^{p_j/2}\sin\theta + o(r_1^5)$$

By virtue of condition

$$4\prod_{j=1}^{m}|B_j|^{p_j/2} \geqslant \left|\sum_{\alpha=1}^{m} A_\alpha B_\alpha\right| \tag{2.11}$$

the derivative $V^{\cdot} > 0$ in domain $V > 0$. By the same token, all the hypotheses of Chetaev's instability theorem /9/ have been fulfilled.

**Theorem 2.** If $B_\alpha \neq 0$ $(\alpha = 1, \ldots, m)$, then a necessary and sufficient stability condition for model system (2.4) is the fulfilment of one of the conditions: a) a pair of coefficients $B_i, B_j (i \neq j)$ exists such that $\operatorname{sign} B_iB_j = -1$; b) $\operatorname{sign} B_iB_j = 1$ and condition (2.8) is valid for all possible pairs of coefficients $B_i, B_j$. However, if all $B_\alpha$ $(\alpha = 1, \ldots, m)$ are of the same sign and (2.11) is fullilled, then the equilibrium position of system (1.1) is Liapunov-unstable.

We remark that all the conclusions in the paper are valid not only for systems (1.1) with position forces but also for any invertible systems in the sense of the well-known definition (see /5/, p. 35). This follows from /4,5/. For example, suppose that forces

$$Q^*_s = \sum_{i,j=1}^{n} f_{sij}(q)\, q_i^{\cdot} q_j^{\cdot} \quad (s = 1, \ldots, n)$$

where $f_{sij}$ are holomorphic functions of $q$, act on system (1.1) in addition to the position forces. Then all the transformations in Sect. 1 remain in force and as a result we arrive at (1.9) and (1.10). Forces of the kind mentioned occur, for example, in nonholonomic Chaplygin systems /10/.

**3. Example. Model of an elastic rod under the action of a tracking force /1/.** One group of stability problems with nonconservative position forces is connected with elastic systems acted on by so-called tracking forces, i.e., forces whose action lines coincide with the tangents to the elastic axis of the rod. We can analyze one mechanical system as a model of these systems (see /1/). Retaining the notation in /1/, we pose the problem on the stability of the equilibrium position $\varphi_1 = \varphi_2 = 0$, solved in the linear approximation. We write the equations of perturbed motion, solved with respect to the highest derivatives, in the form:

$$\varphi_s^{\cdot\cdot} = b_{s1}\varphi_1 + b_{s2}\varphi_2 + \Phi_s + \ldots \quad (s = 1, 2)$$

$$b_{11} = -[(c_1 + c_2 - Fl_1)\,a_{22} + c_2a_{12}]\,/\,\Delta, \quad b_{12} = [(c_2 - Fl_1)a_{22} + c_2a_{12}]\,/\,\Delta$$

$$b_{21} = [(c_1 + c_2 - Fl_1)\,a_{21} + c_2a_{11}]\,/\,\Delta, \quad b_{22} = -[(c_2 - Fl_1)\,a_{21} + c_2a_{11}]\,/\,\Delta$$

$$\Delta\Phi_1 = a_{12}\,(a_{22}\varphi_2^{\cdot} - a_{12}\varphi_1^{\cdot})\,(\varphi_1^{\cdot} - \varphi_2^{\cdot})\,(\varphi_1 - \varphi_2) - \frac{1}{2}\left(c_2a_{12} - \frac{Fl_1a_{22}}{3}\right) \times$$
$$(\varphi_1 - \varphi_2)^3 - a_{12}^2\,(b_{11}\varphi_1 + b_{12}\varphi_2)\,(\varphi_1 - \varphi_2)^2$$

$$\Delta\Phi_2 = a_{12}\,(a_{11}\varphi_1^{\cdot} - a_{12}\varphi_2^{\cdot})\,(\varphi_1^{\cdot} - \varphi_2^{\cdot})\,(\varphi_1 - \varphi_2) - \frac{Fl_1a_{12}}{6}\,(\varphi_1 - \varphi_2)^3 +$$
$$\frac{a_{12}}{2}\,[(c_1 + c_2 - Fl_1 - 2a_{12}b_{21})\,\varphi_1 - (c_2 - Fl_1 + 2a_{12}b_{22})\,\varphi_2]\,(\varphi_1 - \varphi_2)^2$$

$$\Delta = a_{11}a_{22} - a_{12}^2$$

The terms not written out are of order higher than third in $\varphi_s$ and $\varphi_s^{\cdot}$ $(s = 1, 2)$. In order that the characteristic equation

$$\varkappa^4 + a\varkappa^2 + b = 0, \; a = -(b_{11} + b_{22}), \; b = b_{11}b_{22} - b_{12}b_{21}$$

have pure imaginary roots, it is necessary and sufficient to fulfil the conditions

$$a > 0, \; b > 0, \; a^2 - 4b \geqslant 0 \tag{3.1}$$

From these inequalities we can find the least value of tracking force $F$, under which the stability of the system in the linear approximation is preserved. (The system is stable in the absence of the tracking force /1/):

Let us find the system's frequencies

$$\omega_1^2 = \frac{a}{2} + \sqrt{\frac{a^2}{4} - b}, \quad \omega_2^2 = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b}, \quad \omega_1^2 \geqslant \omega_2^2$$

In accord with Sect. 1, if the relations $\omega_1^2 = p^2\omega_2^2$ are not fulfilled for any integers $p \geqslant 1$, then the system is completely Birkhoff-stable. Let us reduce the linear-approximation system to form (1.6). The matrix $\|p_{sj}\|$ has the form

$$p_{s1} = -i\,(b_{22} + \omega_s^2)\,/\,b_{12}, \; p_{s2} = i \quad (s = 1, 2)$$

whence

$$\varphi_1 = \frac{b_{12}}{2\,(\omega_1^2 - \omega_2^2)}\left[\frac{z_1 + \bar{z}_1}{\omega_1} - \frac{z_2 + \bar{z}_2}{\omega_2}\right], \quad \varphi_2 = \frac{1}{2\,(\omega_1^2 - \omega_2^2)} \times \tag{3.2}$$
$$\left[\frac{b_{22} + \omega_2^2}{\omega_1}\,(z_1 + \bar{z}_1) - \frac{b_{22} + \omega_1^2}{\omega_2}\,(z_2 + \bar{z}_2)\right]$$

$$\varphi_1^{\cdot} = \frac{b_{12}i}{2\,(\omega_1^2 - \omega_2^2)}\,[(z_1 - \bar{z}_1) - (z_2 - \bar{z}_2)], \quad \varphi_2^{\cdot} = \frac{i}{2\,(\omega_1^2 - \omega_2^2)} \times$$
$$[(b_{22} + \omega_2^2)\,(z_1 - \bar{z}_1) - (b_{22} + \omega_1^2)\,(z_2 - \bar{z}_2)]$$

As a result of the transformations indicated the system acquires the form

$$z_1^{\cdot} = i\omega_1 z_1 - \frac{b_{22}+\omega_1^2}{b_{12}} i\Phi_1^*(z,\bar{z}) + i\Phi_2^*(z,\bar{z}) + \ldots \quad (3.3)$$

$$z_2^{\cdot} = i\omega_2 z_2 - \frac{b_{22}+\omega_2^2}{b_{12}} i\Phi_1^*(z,\bar{z}) + i\Phi_2^*(z,\bar{z}) + \ldots$$

where $\Phi_1^*$ and $\Phi_2^*$ are the functions $\Phi_1$ and $\Phi_2$ are substitutions (3.2).

Suppose that the resonance $\omega_1 = 3\omega_2$ obtains in the system; for it to be present it is sufficient that the condition

$$9a^2 = 100b > 0 \quad (3.4)$$

be fulfilled. Then, after normalization system (3.3) becomes

$$u_1^{\cdot} = i\omega_1 u_1 + iu_1 (A_{11} |u_1|^2 + A_{12} |u_2|^2) + iB_1 u_2^3 + \ldots \quad (3.5)$$

$$u_2^{\cdot} = i\omega_2 u_2 + iu_2 (A_{21} |u_1|^2 + A_{22} |u_2|^2) + iB_2 u_1 v_2^2 + \ldots$$

The real constants $A_{ij}$ and $B_i$ $(i, j = 1, 2)$ are the coefficients in the right hand sides of (3.3) of the same products of variables as in (3.5) if in (3.3) $z$ and $\bar{z}$ are replaced by $u$ and $v$. Without writing out their unwieldy expressions, we draw the following conclusions of the basis of Theorem 2.

If the following inequalities

$$B_1 B_2 < 0, \quad 4 |B_1 B_2| \geqslant |(A_{11} - 3A_{21}) B_1 - (A_{12} - 3A_{22}) B_2| \sqrt{-B_1/B_2} \quad (3.6)$$

are fulfilled together with condition (3.4), the equilibrium position $\varphi_1 = \varphi_2 = 0$ is unstable. If, however, the second inequality in (3.6) reverses of if $B_1 B_2 > 0$, then stability is guaranteed for the system truncated up to cubic terms.

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